

# Predicate Refinement Heuristics in Program Verification with CEGAR

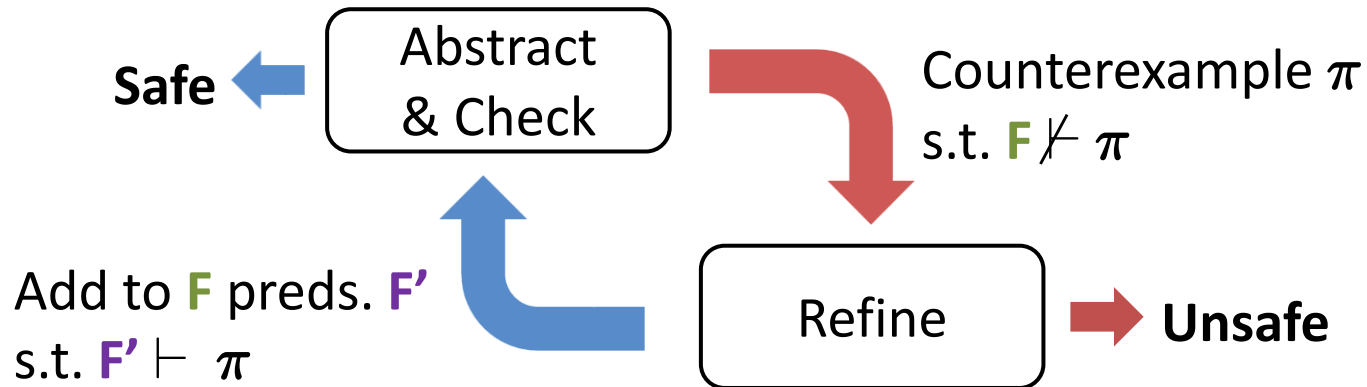
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Part of this is joint work with Hiroshi Unno (U. Tsukuba)

# Predicate Abstraction with CEGAR

Iteratively generate **candidate predicates**  $F \subseteq \text{Preds}(T)$

- until  $F$  forms a proof of the given program
  - $T$  : background FOL theory (e.g., QFLRA)



**Much success for imperative programs (SLAM, BLAST, ...)**  
**for concurrent programs (Threader, SymmPA, ...)**  
**for functional programs (Depcegar, MoCHi, ...)**

# Predicate Refinement

- Input:
  - Currently irrefutable counterexample  $\pi$ 
    - i.e.,  $F \not\vdash \pi$  where  $F$  is current candidate pred. set
- Output:
  - Set of predicates  $F'$  such that  $F' \vdash \pi$

## Issue:

- There can be multiple (in general,  $\infty$  many)  $F'$  s.t.  $F' \vdash \pi$
- Choice of  $F'$  can significantly affect CEGAR performance

# Example

(How refinement choice affects CEGAR performance)

```
a = nondet(); b = nondet();
x = a; y = b; z = 0;
while (nondet()) {
  y++;z++;
}
while (z != 0) {
  y--;z--;
}
if (a=b && y!=x) { assert false; }
```

Proof of the program:

- $\phi_{\text{inv}} \equiv a=b \Rightarrow y=x+z$

Counterexample  $\pi_1$

```
a = nondet(); b = nondet();
x = a; y = b; z = 0;
if (nondet()) {
  y++;z++;
}
if (z != 0) {
  y--;z--;
}
if (a=b && y!=x) { assert false; }
```

Proof of  $\pi_1$ :  $\{ \phi_0, \phi_1, \phi_0 \vee \phi_1 \}$

- $\phi_0 \equiv x=a \wedge y=b \wedge z=0$
- $\phi_1 \equiv x=a \wedge y=b+1 \wedge z=1$

**Sufficient for  $\pi_1$  but not for the program**

# Example

(How refinement choice affects CEGAR performance)

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if (a=b && y!=x) { assert false; }
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Proof of the program:

- $\phi_{inv} \equiv a=b \Rightarrow y=x+z$

$\pi_1$  : refuted by  $\{ \phi_0, \phi_1, \phi_0 \vee \phi_1 \}$   
 $\pi_2$  : refuted by  $\{ \forall F \mid F \subseteq \{ \phi_0, \phi_1, \phi_2 \} \}$   
⋮  
 $\pi_i$  : refuted by  $\{ \forall F \mid F \subseteq \{ \phi_0, \dots, \phi_i \} \}$   
⋮

$\pi_i \equiv$  Loops unfolded  $i$  times

$\phi_i \equiv x = a \wedge y = b + i \wedge z = i$

**CEGAR DIVERGES!**

# Outline

- ✓ Introduction
- 2. Refinement scheme with convergence guarantee
- 3. Fast convergence via “small refinements”

Based on [Terauchi, Unno ESOP 2015]

# **REFINEMENT SCHEME WITH GUARANTEED CEGAR CONVERGENCE**

# Example

(How refinement choice affects CEGAR performance)

```
a = nondet(); b = nondet();
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 $\vdots$   
 $\pi_i$  : refuted by  $\{ \forall F \mid F \subseteq \{ \phi_0, \dots, \phi_i \} \}$   
 $\vdots$

$\pi_i \equiv$  Loops unfolded  $i$  times  
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Proof of the program:

- $\phi_{inv} \equiv a=b \Rightarrow y=x+z$

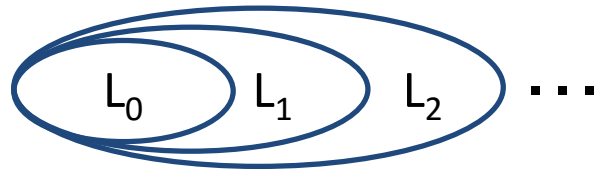
**Key Observation:**  $\phi_{inv}$  refutes every  $\pi_i$

**$\therefore$  Can force convergence by restricting predicates inferred by refinement**



# Stratified Refinement [1,2]

- Prepare growing strata of predicate sets:



- Each  $L_i \subseteq \text{Preds}(T)$  is finite
- $\text{Preds}(T) = \bigcup_{i=1}^{\omega} L_i$
- In each refinement step:
  - Restrict inferred predicates to some  $L_i$
  - raise  $L_i$  to next level when no proof of given c.e.x. is in  $L_i$

## **GUARANTEED CEGAR CONVERGENCE**

- **under promise that a proof exists in T**

[1] R. Jhala, K. McMillan. Practical and complete approach to predicate refinement. TACAS'06.

[2] K. McMillan. Quantified invariant generation using an interpolating saturation prover. TACAS'08.

# Issue with Stratified Refinement

- Refinement step must decide if current  $L_i$  has a proof of given counterexample
  - i.e., decide if  $\exists F \subseteq L_i . F \vdash \pi$
  - Such **exact finite-predicate-set-restricted proof search** is hard
    - cf. ESOP'15 paper for details

## Our Goal

- More Practical Refinement with Convergence Guarantee**
- **under the same promise that a proof exists in  $\text{Preds}(T)$**

# New Proposal: Relaxed Stratification

- Prepare strata of predicate set pairs

$$B_0 \cup E_0, B_1 \cup E_1, \dots, B_i \cup E_i, \dots$$

Base & Extension

- Each  $B_i \cup E_i \subseteq \text{Preds}(T)$  is finite
  - $B_i \subseteq B_{i+1}$  for each  $B_i$
  - $\text{Preds}(T) = \bigcup_{i=1}^{\omega} B_i$
- In each refinement step:
    - Restrict inferred predicates to some  $B_i \cup E_i$
    - Fail to infer preds. and raise level only if no proof is in  $B_i$

(Exact) stratification is  
special case where  $E_i = \emptyset$

Need not to exactly decide existence of proof in  $B_i$  or in  $B_i \cup E_i$

# Correctness

**Theorem:** With relaxed stratification, CEGAR converges under the promise that program can be proved by Preds(T)

**Proof sketch:**

- Follows from **Key Observation:** proof of program is proof of its counterexamples. Therefore:
  - Stratum only goes up to  $B_i \cup E_i$  where  $B_i \supseteq$  proof of prog.
  - Stays in same stratum only for finite number of CEGAR iterations

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- 2. Refinement scheme with convergence guarantee
  - ✓ Relaxed Stratification Scheme
  - b. Concrete instances of relaxed stratification
    - “Restricted” recursion-free Horn-clause solvers
  - c. Experiments
- 3. CEGAR iteration bound via “small” refinement

# Horn Clause Constraints

*Predicate variables*  $P, Q, P_1, P_2, \dots$

*Body clause*  $B ::= \theta \mid P(\vec{x}) \mid B \wedge B$

*Horn clause constraint*  $hc ::= B \Rightarrow P(\vec{x}) \mid B \Rightarrow \perp$

*Horn clause constraint set*  $H ::= \emptyset \mid H \cup \{hc\}$

- $H$  is *satisfiable* if  $\exists \sigma : pvars(H) \mapsto \text{Preds}(\mathcal{T}). \forall hc \in H. \models \sigma(hc)$ .
- $H$  is *recursion-free* if  $\{(P, Q) \mid \dots P(\dots) \dots \Rightarrow Q(\dots) \in H\}$  is acyclic.
- $H$  is *tree-like* if each  $P \in pvars(H)$  occurs at most once in left of  $\Rightarrow$  and at most once in right of  $\Rightarrow$ .

# Reducing Predicate Refinement to Horn-clause Solving

**Prop:** For any  $\pi$ , exists recursion-free  $H_\pi$  such that  
 $\sigma$  is solution of  $H_\pi \iff \{\sigma(P) \mid P \in \text{dom}(\sigma)\} \vdash \pi$

Standard Refinement Algorithm:

1. Build  $H_\pi$
2. Find solution  $\sigma$  of  $H_\pi$
3. Return  $\{\sigma(P) \mid P \in \text{dom}(\sigma)\}$  as inferred predicates

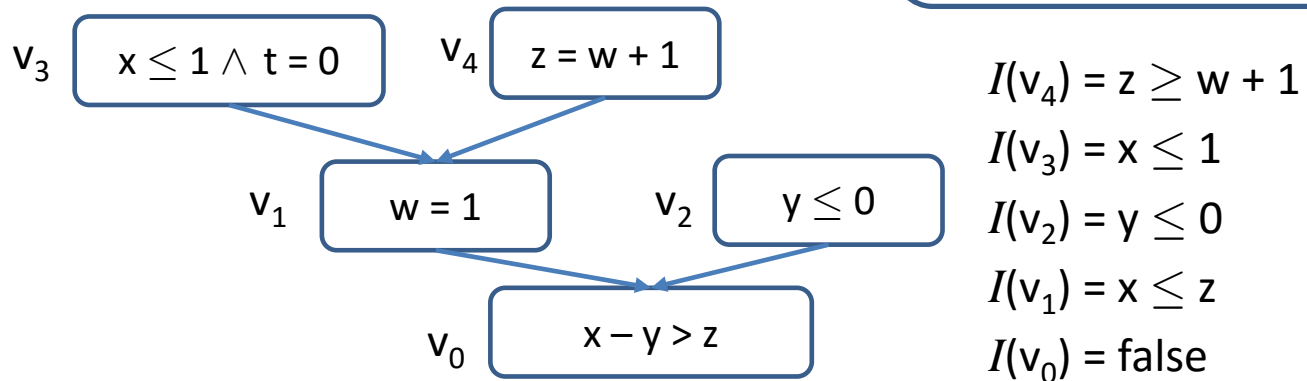
**Plan:** Modify Step 2 so that

- Finds solution from  $B_i \cup E_i$  (**not from entire Preds(T)**)
- Fails to solve only if no solution is in  $B_i$

# Technical Detour: Tree Interpolation

- Labeled tree  $(V, E, \Theta)$  -  $V$ : nodes  $E$ : edges  $\Theta: V \rightarrow \text{Formula}(T)$
- $I: V \rightarrow \text{Formula}(T)$  is **tree interpolant (ITP)** of  $(V, E, \Theta) \Leftrightarrow$ 
  - For root  $v \in V, I(v) = \text{false}$
  - $\forall v \in V. \Theta(v) \wedge (\bigwedge_{(v,v') \in E} I(v')) \Rightarrow I(v)$
  - $\forall v \in V. \text{vars}(I(v)) \subseteq \text{sharedvars}(v)$

vars. occurring both in & out of subtree rooted at  $v$





# Reducing Horn clauses to Tree Interpolation

**Theorem[Rümmer+'13]:** Given tree-like clause set  $H$ , can build tree-interpolation instance  $(V_H, E_H, \Theta_H)$  s.t.

$$\{P \mapsto \lambda \vec{x}_P. I(P)\} \models H \Leftrightarrow I \text{ is tree-interpolant of } (V_H, E_H, \Theta_H)$$

where  $V_H = pvars(H) \cup \{v_{rt}\}$ ,  $v_{rt} \notin pvars(H)$ , and  $\vec{x}_P$ 's are fresh variables.

Can be extended to general recursion-free case (cf. [Rümmer+'13])

– Example:

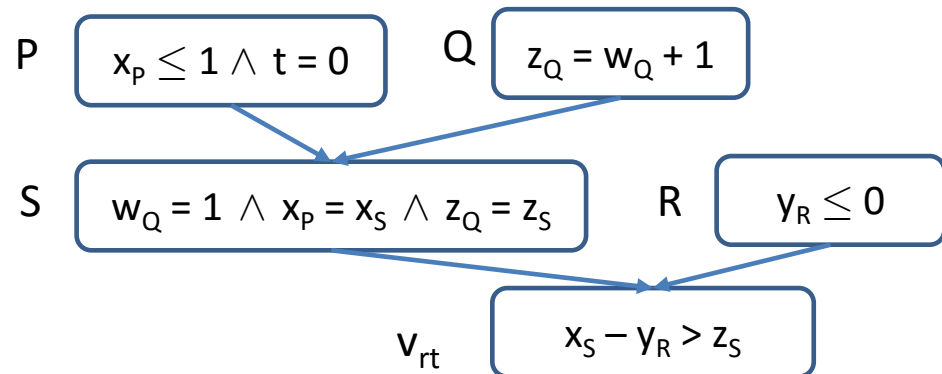
$$x \leq 1 \wedge t = 1 \Rightarrow P(x)$$

$$z = x + 1 \Rightarrow Q(w, z)$$

$$y \leq 0 \Rightarrow R(y)$$

$$w = 1 \wedge P(x) \wedge Q(w, z) \Rightarrow S(x, z)$$

$$x - y > z \wedge S(x, y) \Rightarrow \perp$$



# Review

- We want to:
  - Infer solution  $\sigma$  of  $H_\pi$  s.t.  $\text{ran}(\sigma) \subseteq B_i \cup E_i$
  - Fail to infer only when no solution of range  $B_i$  exists
- Horn-clause to tree interpolation reduction says:  
 $\{P \mapsto \lambda \vec{x}_P. I(P)\}$  is solution of  $H \Leftrightarrow I$  is tree interpolant of  $(V_H, E_H, \Theta_H)$
- So, it suffices to do “restricted” tree interpolation:
  - Infer tree interpolant  $I$  of  $(V_H, E_H, \Theta_H)$  s.t.  $\text{ran}(I) \subseteq B_i \cup E_i$
  - Fail to infer only when no interpolant of range  $B_i$  exists

# Restricted Tree Interpolation

## Standard tree interpolation algorithm:

- Input:  $(V, E, \Theta)$
- Use SMT solver to check  $\bigwedge_{v \in V} \Theta(v)$  is UNSAT
  - obtain resolution proof deriving “false”
- Compute **partial ITPs** at each node of resolution proof
  - ITP = partial ITP at root node

## Modification: theory-level-restricted SMT solving

- Restrict leaf (i.e., theory) level reasoning to only produce partial ITPs in  $B_i$ 
  - Prototype implementation using template-based technique

**Only uses expensive finite preds. res. search at leaf levels**

# Tree Interpolant Generation

- $p \in \text{Atoms}(T)$
- $C ::= p \mid \neg p \mid C \vee C$

$$\frac{C \wedge \dots = \Theta(v')}{(V, E, \Theta) \vdash C : I}$$

$$I = \lambda v. \begin{cases} \text{false} & \text{if } (v', v) \in E^* \\ \text{true} & \text{otherwise} \end{cases}$$

$$\frac{(V, E, \Theta) \vdash p \vee C_1 : I_1 \quad (V, E, \Theta) \vdash \neg p \vee C_2 : I_2}{(V, E, \Theta) \vdash C_1 \vee C_2 : I}$$

$$I = \lambda v. \begin{cases} I_1(v) \wedge I_2(v) & \text{if } p \in \text{outs}(v) \\ I_1(v) \vee I_2(v) & \text{otherwise} \end{cases}$$

$$\frac{\models_T C}{(V, E, \Theta) \vdash C}$$



$$\frac{I \text{ is tree itp. of } (V, E, \Theta_C) \text{ where } \text{ran}(I) \subseteq B_i}{(V, E, \Theta) \vdash C : I}$$

$$\Theta_C = \Theta \text{ with labels restricted to } \text{Atoms}(C)$$

**Theorem:** Inferred ITP is in  $B_i^{\wedge \vee}$ , and some ITP is inferred if one exists in  $B_i$

So,  $E_i = B_i^{\wedge \vee}$

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- 2. Refinement scheme with convergence guarantee
  - ✓ Relaxed Stratification Scheme
  - b. Concrete instances of relaxed stratification
    - “Restricted” recursion-free Horn-clause solvers
      - ✓ Theory-level-restricted tree interpolation
      - ii. Another concrete instance
  - c. Experiments
- 3. CEGAR iteration bound via “small” refinement

# Another Concrete Instance of Relaxed Stratification Scheme

- “Restricted” Horn clause constraint solver (again)

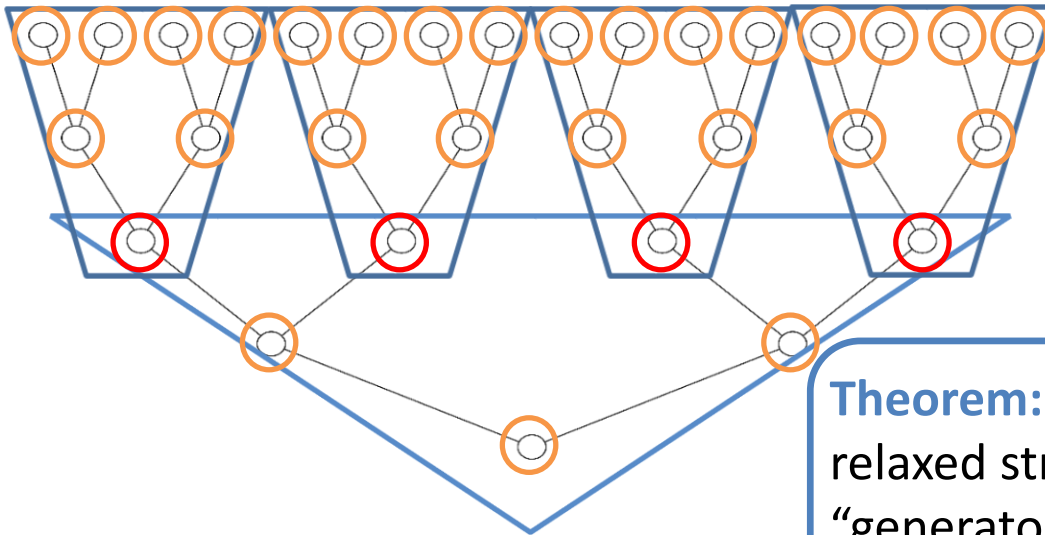
## Two-phase approach

1. Partition clauses into bounded-size trees
2. Infer restricted solutions to predicate variables at partition boundaries
3. Infer unrestricted solutions to the rest

**Only use expensive finite predicate-restricted search for few predicate variables**

**Becomes relaxed stratification under some mild assumptions on generated counterexample patterns**

# Algorithm Explained Pictorially



**Theorem:** This satisfies the requirements of relaxed stratification, assuming there is a “generator” clause set  $H_{\text{gen}}$  s.t. any c.ex. is an unfolding of  $H_{\text{gen}}$

- 1. Partition into bounded-size sub-trees**
- 2. Infer restricted solutions to boundary predicate variables**
  - E.g., via template-based method
    - Could also use another relaxed stratification refinement alg. itself
- 3. Infer unrestricted solutions to the rest**
  - via standard approaches [Unno+'09, Terauchi'10, Rümmer+'13, etc.]

# Refinement Algorithm Schemas

- These are actually **algorithm schemas**
  - take other refinement algorithms as modules
  - generate refinement algorithms satisfying requirements of relaxed stratification

**See ESOP'15 paper for details**



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      - ✓ Theory-level-restricted tree interpolation
      - ✓ 2-phase inference approach via bounded partitioning
  - c. Experiments
- 3. Fast convergence via “small refinements”

# Prototype Implementation

## New refinement algorithm

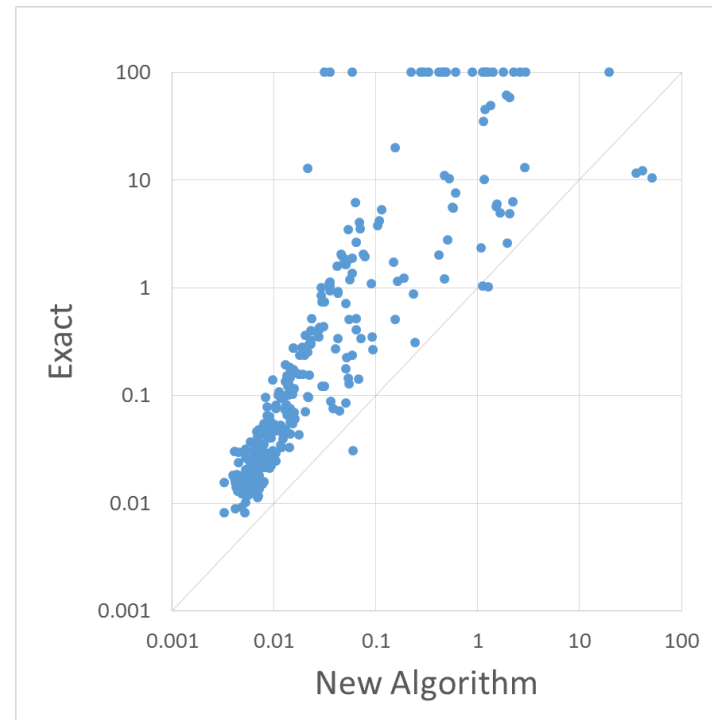
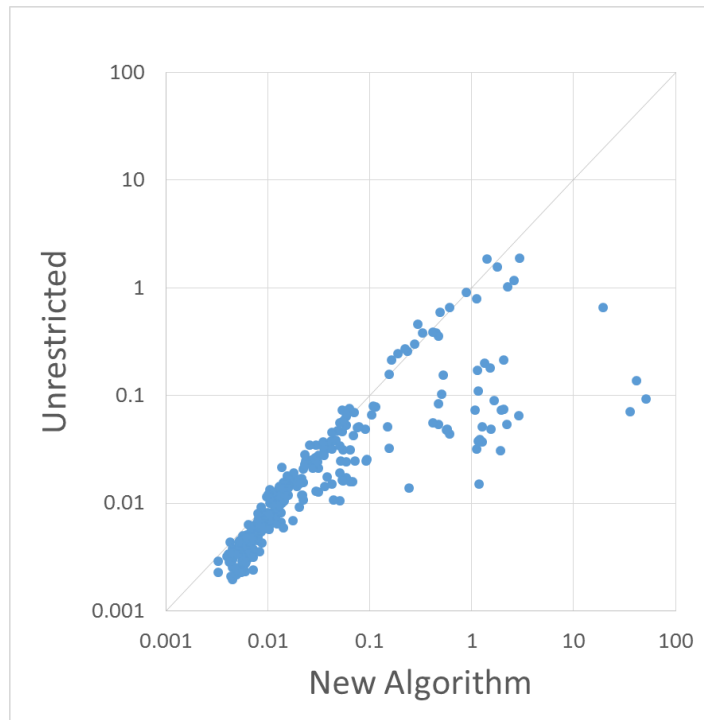
- Algorithm 1 used as module of algorithm 2
- Z3 [1] for template-based constraint solving
- MathSAT5 [2] for unrestricted refinement
- Used as refinement engine of MoChi [3]
  - Software model checker for higher-order functional programs based on CEGAR

[1] <http://z3.codeplex.com/>

[2] <http://mathsat.fbk.eu/>

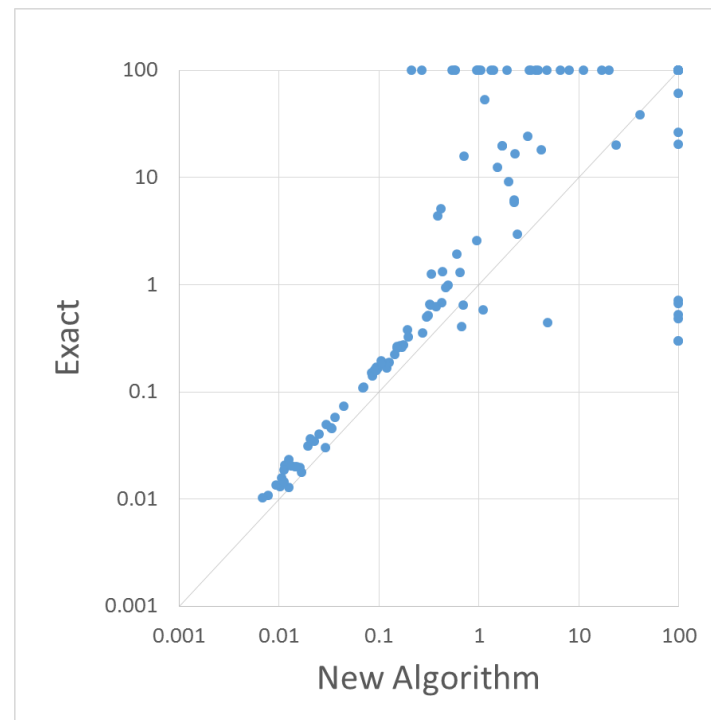
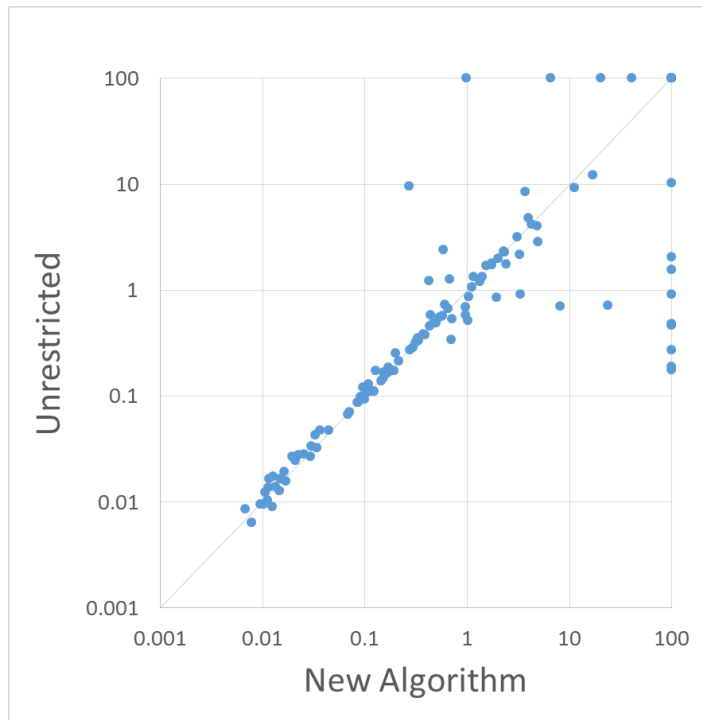
[3] <http://www.kb.is.s.u-tokyo.ac.jp/~ryosuke/mochi/>

# Experiment Results: Individual Refinement Runs



- 318 counterexamples generated from 139 benchmark programs
- Three refinement algorithms:
  - New algorithm
  - Unrestricted refinement
  - Exact stratification

# Experiment Results: Overall Verification Performance



- 139 benchmark programs
- MoChi with each refinement algorithm:
  - New algorithm
  - Unrestricted refinement
  - Exact stratification

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  - ✓ Relaxed Stratification Scheme
  - ✓ Concrete instances of relaxed stratification
    - ✓ “Restricted” Horn clause solvers algorithms
  - ✓ Experiments
- 3. Fast convergence via “small refinements”

Based on [Terauchi SAS 2015]

# **FAST CONVERGENCE VIA “SMALL REFINEMENTS”**

# Talk so far

## Predicate refinement in CEGAR

- Return predicates that refutes given counterexample
- Clever choice of predicates can make CEGAR converge

**Q: Can we say anything about convergence speed?**

**Short Answer: YES**

# Our Result

## Small Refinement Heuristic (SRH)

- Refinement phase returns “small” proof of counterexample’s safety

[Hoder+’12][Scholl+’14][Albarghouthi,McMillan’14][Unno,Terauchi’15]

- **We will show:**

- CEGAR with SRH converges in number of CEGAR iterations bounded in the size of the proof for the input program



# Example (from before)

```
a = nondet(); b = nondet();
x = a; y = b; z = 0;
while (nondet()) {
  y++;z++;
}
while (z != 0) {
  y--;z--;
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if (a=b && y!=x) { assert false; }
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$\pi_i \equiv$  Loops unfolded  $i$  times  
 $\phi_i \equiv x = a \wedge y = b + i \wedge z = i$

Proof of the program:

$\phi_{\text{inv}} \equiv a=b \Rightarrow y=x+z$

**Key Observation:**  $\phi_{\text{inv}}$  refutes every  $\pi_i$

How to define “small”?

**$\therefore$  Inferring small refinements should hasten convergence**  
Refinement will infer  $\phi_{\text{inv}}$  or some other small proof of program before inferring large  $\phi_k$ 's

# Proof Size Metric and SRH

**Def:**  $\text{size} : \mathcal{P}_{\text{fin}}(\text{Preds}(T)) \rightarrow \text{Nat}$  is **generic proof size metric** if  $\exists c > 0. \forall n \geq 0. |\{ F \mid \text{size}(F) \leq n \}| \leq c^n$

**Def:**  $\text{minprfsize}(\gamma) = \min_{F \in \{F \mid F \vdash \gamma\}} \text{size}(F)$

$\gamma$  : cex or program

**Def:** CEGAR with **Small Refinement Heuristic (SRH)** is CEGAR with  $\text{Refine}()$  satisfying:

$\exists \text{poly } f. \forall \pi. \text{if } \text{Refine}(\pi) = F \text{ then } \text{size}(F) \leq f(\text{minprfsize}(\pi))$

# Convergence Bound

**Theorem:** Suppose proof size metric is generic. Then CEGAR with SRH converges in number of iterations bounded exponentially in  $\text{minprfsize}(P)$ .

– **Proof:** By **KEY OBSERVATION** and simple counting argument

**Corollary:** CEGAR with SRH converges in exponential number of iterations under the promise that program has polynomial size proof

Can a tighter bound be obtained with a more concrete setting?

# Bound for CFG-represented Programs

- Assumptions
  - Program represented by reducible Control Flow Graph
  - Counterexamples are loop-unfoldings of the CFG
    - every loop unfolded the same number of times
  - Proof is Floyd-style node-wise inductive invariant
  - Predicate abstraction is Cartesian predicate abstraction
  - $\text{size}(F) = \sum_{\phi \in F} \text{syntactic\_size}(\phi)$
- **Theorem: CEGAR with SRH converges in number of iterations bounded polynomially in  $\text{minprfsize}(P)$  for CFG programs.**

See SAS'15 paper for details

# Outline

- ✓ Introduction
- ✓ Refinement scheme with convergence guarantee
  - ✓ “Relaxed” Stratification
- 3. Fast convergence via “small refinements”
  - ✓ Generic Setting
    - ✓  $\exp(\text{minprfsize}(P))$  CEGAR iteration bound
  - ✓ CFG-represented Programs
    - ✓  $\text{poly}(\text{minprfsize}(P))$  CEGAR iteration bound (under various assumptions)

# Some Thoughts and Future Work

## **Results on CEGAR iteration bound can be taken as negative results?**

- Inferring small refinements must be hard because otherwise verification would be easy?
  - Substantiates the experience with stratified refinement

## **Need further investigation**

- Hardness of verification w.r.t. “predicate refinement oracles” and under “small proof promise” seems to be underexplored

